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The Runaway Effect in a Lorentz Gas

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The Lorentz gas of charged particles in a constant and uniform electric field is studied. The gas flows through the medium of immobile, randomly distributed scatterers. Particles with velocity $\mathbf{v}$ suffer collisions with frequency proportional to $|\mathbf{v}|^n$. For $n < 0$ runaway of the gas is forced by the field: the mean velocity of the flow increases without bounds. By a simple physical argument an integral relation is established between the probability of collisionless motion and the velocity distribution. It is then shown that when $n < -1$ a fraction of particles moves as if the scattering centers were absent. The detailed discussion of this uncollided runaway is presented. Some qualitative features of the velocity distribution are illustrated on rigorous solutions in one dimension.

KEY WORDS: Lorentz gas; Boltzmann equation; probability density; runaway of particles.

1. INTRODUCTION

The description of runaway electrons in ionized gases under the action of an applied electric field was first given by Giovanelli[1] in 1949. It can be found nowadays in monographs on the kinetic theory of gases.[2] The runaway process results from the lack of balance between the loss of electron energy at encounters with ions and the gain of energy absorbed from the external field. For sufficiently strong fields the electron gas is heated without bounds and acquires large drift velocity, steadily increasing in time. This phenomenon, known as electron runaway, was given an exhaustive discussion by Dreicer[3] in the case of fully ionized gases (see also Ref. 4). Dreicer remarked the special role played by rapid particles whose interaction with ions is negligibly small. Motion of these particles is

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practically free from encounters. So they run away in arbitrarily weak fields.

Runaway phenomena for electrons in neutral gases have also been studied in detail. Relevant references can be found in a paper by Cavalleri and Paveri-Fontana.[5]

We consider here a simple generalization of the classical Lorentz model of the electron gas in metals.[6] The probability density $f(\mathbf{r}, \mathbf{v}, t)$ for finding a particle with velocity $\mathbf{v}$ and position $\mathbf{r}$ at time t is supposed to satisfy a kinetic equation of the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial}{\partial \mathbf{v}}\right) f(\mathbf{r}, \mathbf{v}, t) = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \tag{1.1}$$

where

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = |\mathbf{v}|^n \Lambda^{-1}(\hat{\Pi}_{\mathbf{v}} - 1) f(\mathbf{r}, \mathbf{v}, t) \tag{1.2}$$

The projector $\hat{\Pi}_{\mathbf{v}}$ averages f over all directions in velocity space

$$\hat{\Pi}_{\mathbf{v}} f(\mathbf{r}, \mathbf{v}, t) = \frac{1}{4\pi} \int d\Omega_{\mathbf{v}}\, f(\mathbf{r}, \mathbf{v}, t) \tag{1.3}$$

($d\Omega_{\mathbf{v}}$ is an element of solid angle), $\mathbf{a}$ is the acceleration due to a time-independent, uniform electric field, and Λ is a constant. It is thus assumed that the mutual interaction between the particles can be neglected, and that they are elastically and isotropically scattered in their motion by immobile (infinitely heavy) scattering centers. The structure of collision term (1.2) implies that the collision frequency (number of collisions per unit time) for a particle having the velocity $\mathbf{v}$ is equal to $|\mathbf{v}|^n/\Lambda$. When $n = 1$, Eq. (1.1) becomes the Boltzmann equation studied by Lorentz.[6] The linear growth of collision frequency with $|\mathbf{v}|$ corresponds to the hard-sphere interaction. For $n \neq 1$ the interpretation of the collision term is more subtle. One knows that within the framework of classical mechanics the differential cross section for elastic scattering in the potential field $V(|\mathbf{r}|) \sim |\mathbf{r}|^{-\alpha}$ is proportional to $|\mathbf{v}|^{-4/\alpha}$, where $\mathbf{v}$ is the initial velocity of the incident particle.[7] It is only in this sense that the collision term in Eq. (1.1) can be associated with the interaction $|\mathbf{r}|^{-\alpha}$, with α related to n by

$$1 - 4/\alpha = n \tag{1.4}$$

One should, however, keep in mind that the actual angle dependence of the cross section has been replaced in Eq. (1.1) by that corresponding to the hard-sphere scattering (projector $\hat{\Pi}_{\mathbf{v}}$), which makes this association rather loose. It is thus not surprising that for physical applications other collision terms have been used. For instance, the case of an ideal Lorentz plasma ($\alpha = 1, n = -3$) has been studied with the use of the Fokker–Planck expression for the effect of the small-angle Coulomb collisions.[8] However, our object here is to analyze the appearance of runaway particles, and from this point of view the fundamental role is played by the velocity dependence of the collision frequency, which enters correctly into Eq. (1.1).

Let us define the meaning of the runaway effect in the Lorentz gas in a precise way. We shall say that runaway occurs if the mean particle velocity

$$\langle \mathbf{v} \rangle_t = \int d\mathbf{v}\, f(\mathbf{v}, t)\mathbf{v} \tag{1.5}$$

grows without bounds when $t \to \infty$.

Suppose that a particle has velocity $\mathbf{v}_0$ at time $t = 0$. Let $P(t; \mathbf{v}_0, n)$ denote the probability of finding it with velocity $\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t$ at time t. If

$$\lim_{t \to \infty} P(t; \mathbf{v}_0, n) = P(\infty; \mathbf{v}_0, n) > 0 \tag{1.6}$$

a fraction $P(\infty; \mathbf{v}_0, n)$ of particles, having initially the velocity $\mathbf{v}_0$, moves with acceleration $\mathbf{a}$ as if the scattering centers were absent. The presence of these "uncollided" particles is a sufficient condition for runaway. It is, however, by no means a necessary one. The unlimited growth of $\langle \mathbf{v} \rangle_t$ can occur even when $P(\infty; \mathbf{v}_0, n) = 0$. Indeed, it follows from our previous work[9] (in collaboration with E. Wajnryb) that

$$P(\infty; \mathbf{v}_0, n) = 0 \qquad \text{for} \quad n > -1 \tag{1.7}$$

In this case the mean velocity follows the power law

$$\langle \mathbf{v} \rangle_t \underset{t \to \infty}{\sim} \mathbf{a} t^{-n/(n+2)} \tag{1.8}$$

and thus increases indefinitely for $-1 < n < 0$ [Eq. (1.8) is a direct consequence of Eq. (60) of Ref. 9]. We conclude that particles run away in our model when $n < 0$.

It is interesting to remark that the presence (or absence) of uncollided particles is closely related with symmetry properties of the velocity distribution in the long-time regime. It has been demonstrated in Ref. 9 that the dominant term in the long-time expansion of the velocity distribution has the spherical symmetry, provided $n > -1$. In other words, the collisions are efficient enough to make $f(\mathbf{v}, t)$ asymptotically isotropic despite the presence of the field. This statement holds in any dimension of velocity space. We illustrate it in Section 2, where the rigorous solution of the one-dimensional version of Eq. (1.1) for $n = 0$ is presented. In this sense the whole velocity distribution is collision-dominated even for $n < 0$, provided $n > -1$.

The method of the long-time asymptotic expansion used in Ref. 9 breaks down when $n \leqslant -1$. One can expect that for collision frequencies decreasing sufficiently rapidly with growing velocities the influence of the field on fast particles can introduce an anisotropy persisting even for

$t \to \infty$. This phenomenon will be shown to occur for $n = -1$ in Section 3. When $n < -1$ a new qualitative change appears: condition (1.6) is satisfied for particles with nonzero initial velocities. The appearance of uncollided runaway particles is discussed in detail in Section 4, where an integral relation is established between the probability $P(t; \mathrm{v}_0, n)$ and the solution of the initial value problem $f(\mathrm{v}, t = 0) = \delta(\mathrm{v} - \mathrm{v}_0)$ [see Eq. (4.6)]. This relation is crucial for the rest of the paper. When combined with Eq. (1.1) it yields a rigorous expression for $P(t; \mathrm{v}_0, n)$, which agrees with that used in Ref. 5. The paper ends with a number of comments, including the discussion of the special case $n = -2$, and of the applicability of the conclusions derived for the Lorentz gas to more realistic systems.

2. EXAMPLE OF ASYMPTOTIC SYMMETRY: RIGOROUS SOLUTION FOR $n = 0$ IN ONE DIMENSION

The one-dimensional version of Eq. (1.1) for the velocity distribution $f(v, t)$ has the form

$$\left(\frac{\partial}{\partial t} + a \frac{\partial}{\partial v} \right) f(v, t) = -|v|^n \Lambda^{-1} f^a(v, t) \tag{2.1}$$

where

$$f^a(v, t) = \tfrac{1}{2} \left[f(v, t) - f(-v, t) \right] \tag{2.2}$$

and $|v|$ denotes the modulus of v (velocity has now two possible directions). The symmetric part of f is defined by

$$f^s = f - f^a \tag{2.3}$$

In order to solve the initial value problem

$$f(v, 0) = \delta(v - v_0) \tag{2.4}$$

we use the Laplace transform

$$\tilde{f}(v, z) = \int_0^\infty dt\, e^{-zt} f(v, t), \qquad \mathrm{Re}\, z > 0 \tag{2.5}$$

and rewrite Eq. (2.1) as

$$\left(z + a \frac{\partial}{\partial v} \right) \tilde{f}(v, z) + |v|^n \Lambda^{-1} \tilde{f}^a(v, z) = \delta(v - v_0) \tag{2.6}$$

Equation (2.6) is equivalent to the system of equations

$$z \tilde{f}^s(v, z) + a \frac{\partial}{\partial v} \tilde{f}^a(v, z) = \frac{1}{2} \left[\delta(v - v_0) + \delta(v + v_0) \right] \tag{2.7a}$$

$$(z + |v|^n \Lambda^{-1}) \tilde{f}^a(v, z) + a \frac{\partial}{\partial v} \tilde{f}^s(v, z) = \frac{1}{2} \left[\delta(v - v_0) - \delta(v + v_0) \right] \tag{2.7b}$$

which lead to the second-order equation for the antisymmetric part of $\tilde{f}$,

$$\left[\frac{\partial^2}{\partial v^2} - \frac{1}{a^2}\left(z^2 + z|v|^n\Lambda^{-1}\right)\right]\tilde{f}^a(v,z)$$
$$= -\frac{z}{\Lambda a^2}\left[\delta(v-v_0) - \delta(v+v_0)\right] + \frac{1}{2a}\frac{\partial}{\partial v}\left[\delta(v-v_0) + \delta(v+v_0)\right] \tag{2.8}$$

Equation (2.8) is particularly simple when the collision frequency is constant ($n = 0$). This happens for the $|\mathbf{r}|^{-4}$ (Maxwell) interaction and has been widely exploited in kinetic theory (see, e.g., Ref. 2, p. 173). The general solution of the associated homogeneous equation in this case is a linear combination of functions $\exp(\pm v\sigma)$, where

$$\sigma = (1/a)(z^2 + z/\Lambda)^{1/2} \tag{2.9}$$

For $v > 0, v_0 \neq 0$, the physically relevant solution $\tilde{f}(v,z\,|\,v_0)$ must satisfy the conditions

$$\lim_{v\to\infty} \tilde{f}^a(v,z\,|\,v_0) = \lim_{v\searrow 0} \tilde{f}^a(v,z\,|\,v_0) = 0 \tag{2.10}$$

and consequently have the form

$$\tilde{f}^a(v,z\,|\,v_0) = C_1\theta(|v_0| - v)\operatorname{sh}(v\sigma) + C_2\theta(v - |v_0|)\exp(-v\sigma) \tag{2.11}$$

where θ denotes the Heaviside step function

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{2.12}$$

Inserting formula (2.11) into Eq. (2.8) and using the relation

$$\frac{\partial}{\partial x}\theta(x - x_0) = \delta(x - x_0) \tag{2.13}$$

we get a system of linear equations for the coefficients C_1, C_2. Solving it, we find

$$\tilde{f}^a(v,z\,|\,v_0) = \frac{1}{2a}\left\{\theta(|v_0| - v)\left[\frac{z}{a\sigma}\operatorname{sgn} v_0 - 1\right]e^{-\sigma|v_0|}\operatorname{sh}(v\sigma)\right.$$
$$\left. + \theta(v - |v_0|)\left[\operatorname{ch}(v_0\sigma) + \frac{z}{a\sigma}\operatorname{sh}(v_0\sigma)\right]e^{-\sigma v}\right\} \tag{2.14}$$

Equation (2.14), valid for $v > 0$, suffices to find the function $\tilde{f}^a(v,z\,|\,v_0)$ for any value of velocity because of its antisymmetry. Using then Eq. (2.7a), we get the symmetric part $\tilde{f}^s(v,z\,|\,v_0)$. The formula for the Laplace transform of the complete distribution then follows and we get

$$\tilde{f}(v,z\,|\,v_0) = \varphi(v,z\,|\,v_0) + \varphi(-v_0,z\,|\,-v) \tag{2.15}$$

where

$$\varphi(v,z\,|\,v_0)=\frac{1}{2a}\theta(|v|-|v_0|)\left(\operatorname{sgn}v+\frac{a\sigma}{2}\right)$$
$$\times\left\{\frac{z}{a\sigma}\operatorname{sh}(v_0\sigma)+\operatorname{ch}(v_0\sigma)\right\}\exp(-|v|\sigma)$$

[σ is given by Eq. (2.9)].

It turns out that the calculation of the inverse Laplace transform of $\tilde{f}$ is possible in a closed form due to the relation[10]

$$\frac{1}{(z^2-\alpha^2)^{1/2}}\exp\left[-\beta(z^2-\alpha^2)^{1/2}\right]$$
$$=\int_0^\infty dt\,e^{-zt}\theta(t-\beta)I_0\left(\alpha(t^2-\beta^2)^{1/2}\right),\qquad \beta>0 \tag{2.16}$$

The standard notation I_ν for the modified Bessel functions has been used here (see Ref. 10, p. 371). Rather lengthy calculations lead eventually to the following result:

$$f(v,t\,|\,v_0)=\left(\exp\frac{-t}{2\Lambda}\right)\Bigg[\delta(at+v_0-v)$$
$$+\theta(at-|v-v_0|)\left(\frac{at+v-v_0}{at-v+v_0}\right)^{1/2}$$
$$\times I_1\left(\frac{\left[a^2t^2-(v-v_0)^2\right]^{1/2}}{2a\Lambda}\right)$$
$$+\theta(at-|v+v_0|)I_0\left(\frac{\left[a^2t^2-(v+v_0)^2\right]^{1/2}}{2a\Lambda}\right)\Bigg] \tag{2.17}$$

From Eq. (2.17) we readily deduce that

$$P(t;v_0,0)=\exp(-t/2\Lambda) \tag{2.18}$$

Hence the probability of collisionless motion vanishes exponentially. The characteristic time $\tau=2\Lambda$ has the meaning of mean free time. The peculiar character of the $|\mathbf{r}|^{-4}$ interaction shows up in that $P(t;v_0,0)$ depends neither on the field nor on the initial velocity v_0.

Using the asymptotic formulas

$$I_1(x),\quad I_0(x)\underset{x\to\infty}{\sim}e^x/(2\pi x)^{1/2} \tag{2.19}$$

we deduce that in the long-time limit $[t \gg 2\Lambda,\ t \gg |(v \pm v_0)/a|]$ the probability density $f(v, t \mid v_0)$ takes the form

$$f^{\mathrm{AS}}(v, t \mid v_0) = \frac{1}{4a(\pi\Lambda t)^{1/2}}\left[\exp\left(\frac{-(v-v_0)^2}{4\Lambda a^2 t}\right) + \exp\left(\frac{-(v+v_0)^2}{4\Lambda a^2 t}\right)\right] \tag{2.20}$$

Putting $v_0 = 0$ (or assuming that $v \gg v_0$), we get a simple diffusion law, recovering the result of Ref. 9. The f^{AS} is a symmetric function of velocity.

According to the introductory discussion of Section 1, the case $n = 0$ separates the region in which $\lim_{t\to\infty}\langle v\rangle_t = 0$ (no runaway) from that where $\lim_{t\to\infty}\langle v\rangle_t = \infty$ (runaway occurs). It is thus interesting to calculate the mean velocity for $n = 0$. Using formula (2.17), we evaluate the integral (1.5), obtaining

$$\langle v\rangle_t = \Lambda a + (v_0 - \Lambda a)e^{-t/\Lambda} \tag{2.21}$$

The asymptotic drift velocity is here constant. An analogous result also holds in higher dimensions, where one simply finds $\lim_{t\to\infty}\langle \mathbf{v}\rangle_t = \Lambda\mathbf{a}$ [see Ref. 9, Eq. (60)].

3. APPEARANCE OF ASYMMETRY: $n = -1$

In this section we show that when the collision frequency is proportional to $|\mathbf{v}|^{-1}$ the applied field introduces an anisotropy which persists in the long-time limit. In order to deal with an explicit and simple solution we shall again restrict our considerations to the one-dimensional case. It is worth mentioning that a number of important physical characteristics of the Lorentz gas do not depend on dimension. For instance, exponents of the asymptotic power laws followed by the moments of the velocity distribution have such a property (see Ref. 9). In order to even further simplify the problem, we shall study the case of particles which are initially at rest,

$$f(v, 0) = \delta(v) \tag{3.1}$$

The antisymmetric part of the Laplace transform $\tilde{f}(v, z)$ then satisfies the equation

$$\left[\frac{\partial^2}{\partial v^2} - \left(\frac{z^2}{a^2} + \frac{z}{\Lambda a^2|v|}\right)\right]\tilde{f}^a(v, z) = \frac{1}{a}\frac{d}{dv}\delta(v) \tag{3.2}$$

obtained from Eq. (2.8) by putting $n = 1$, $v_0 = 0$. Introducing the complex variable $x = 2vz/a$, we find that in the region $v > 0$ the associated homogeneous equation takes the form of the normalized Whittaker equation (see

Ref. 11, p. 1059)

$$\left[\frac{d^2}{dx^2} - \frac{1}{4} + \frac{\lambda}{x} + \frac{1/4 - \mu^2}{x^2} \right] W(x) = 0 \tag{3.3}$$

with $\mu = 1/2$, $\lambda = -1/2a\Lambda$. The integrability condition here chooses the Whittaker function $W_{\lambda,1/2}(x)$, so that

$$\tilde{f}^a(v, z \mid 0) = C W_{-1/2a\Lambda, 1/2}(2vz/a) \qquad \text{for} \quad v > 0 \tag{3.4}$$

Equation (3.2) requires that $\tilde{f}^a(v, z \mid 0)$ have a finite jump at $v = 0$

$$\lim_{v \searrow 0} \tilde{f}^a(v, z \mid 0) = 1/2a \tag{3.5}$$

Condition (3.5) combined with the formula

$$W_{-1/2a\Lambda, 1/2}(0) = 1/\Gamma(1 + 1/2a\Lambda) \tag{3.6}$$

fixes the constant C in Eq. (3.4). The relation

$$x^{-1} W_{\lambda,1/2}(x) = \int_0^\infty dt\, \theta\left(t - \frac{1}{2}\right) e^{-xt} \left[\frac{2t-1}{2t+1} \right]^{-\lambda} \Big/ \Gamma(1 - \lambda) \tag{3.7}$$

allows us to calculate the inverse Laplace transform of $\tilde{f}^a$, and we find

$$f^a(v, t \mid 0) = \frac{1}{2a} \theta(at - |v|) \operatorname{sgn} v \, \frac{\partial}{\partial t} \left[\frac{at - |v|}{at + |v|} \right]^{1/2a\Lambda} \tag{3.8}$$

Inserting this result into Eq. (2.1) with $n = 0$, we readily deduce the form of the complete distribution

$$f(v, t \mid 0) = \frac{\theta(at - |v|)}{2a\Lambda} \, \frac{1}{at - v} \left[\frac{at - |v|}{at + |v|} \right]^{1/(2a\Lambda)} \tag{3.9}$$

It can be checked that in the sense of distributions

$$\lim_{t \searrow 0} f(v, t \mid 0) = \delta(v) \tag{3.10}$$

Denoting by P_+ and P_- the probabilities of finding a particle with positive and negative velocity, respectively, we find

$$P_\pm = \frac{1}{2a\Lambda} \int_0^1 du \, \frac{1}{1 \mp u} \left[\frac{1-u}{1+u} \right]^{1/(2a\Lambda)} \tag{3.11}$$

The obvious inequality $P_+ > P_-$ reflects the asymmetry imposed by the electric field (probabilities P_+, P_- do not depend on time).

Let us remark that the density $f(v, t \mid 0)$ has the homogeneity property

$$f(\alpha v, \alpha t \mid 0) = \alpha^{-1} f(v, t \mid 0), \qquad \alpha > 0 \tag{3.12}$$

It follows that kth moment of f is proportional to t^k. In particular, the mean velocity grows linearly with time. However, it would be erroneous to infer that a fraction of particles moves with a constant acceleration. Indeed, there is no term in Eq. (3.9) proportional to $\delta(at - v)$, so that the probability of collisionless motion is zero at any time! This peculiar behavior is due to the $|v|^{-1}$ divergence of the collision frequency for $v \to 0$. As we shall see in the next section, the situation is quite different for particles with nonzero initial velocities. However, in all cases the probability of a motion without collisions vanishes as $t \to \infty$; the uncollided runaway effect is absent.

4. RUNAWAY WITHOUT COLLISIONS: $n < -1$

In this section we derive the formula for the probability $P(t; \mathbf{v}_0, n)$ without restrictions on the dimension of the space. We then show that when $n < -1$ uncollided runaway takes place.

Our considerations will be based on the kinetic equation

$$\left\{ \frac{\partial}{\partial t} + \mathbf{a} \cdot \frac{\partial}{\partial \mathbf{v}} + |\mathbf{v}|^n \Lambda^{-1}(1 - \hat{\Pi}_{\mathbf{v}}) \right\} f(\mathbf{v}, t) = 0 \tag{4.1}$$

Due to energy conservation, a knowledge of its solution suffices to deduce the complete distribution $f(\mathbf{r}, \mathbf{v}, t)$ satisfying Eq. (1.1), provided the initial position space inhomogeneity occurs only in the direction of the field (see Ref. 9). Let us denote by $f(\mathbf{v}, t \,|\, \mathbf{v}_0)$ the probability density satisfying Eq. (4.1) and the initial condition

$$f(\mathbf{v}, 0 \,|\, \mathbf{v}_0) = \delta(\mathbf{v} - \mathbf{v}_0) \tag{4.2}$$

If the probability of collisionless motion during time interval t is $P(t; \mathbf{v}_0, n)$, the density $f(\mathbf{v}, t \,|\, \mathbf{v}_0)$ contains a contribution of the form

$$P(t; \mathbf{v}_0, n)\, \delta(\mathbf{a}t + \mathbf{v}_0 - \mathbf{v}) \tag{4.3}$$

On the other hand, the rate at which particles suffer first collisions at a given moment τ, $0 \leqslant \tau \leqslant t$, is given by

$$-\frac{\partial}{\partial \tau} P(\tau; \mathbf{v}_0, n) \tag{4.4}$$

The particle velocity after the first collision has modulus $|\mathbf{a}\tau + \mathbf{v}_0|$ and an arbitrary direction $\mathbf{u}$, $|\mathbf{u}| = 1$. It follows that the contribution to $f(\mathbf{v}, t \,|\, \mathbf{v}_0)$ from particles colliding first at time τ has the form

$$\left[-\frac{\partial}{\partial \tau} P(\tau; \mathbf{v}_0, n) \right] \hat{\Pi}_{\mathbf{u}} f(\mathbf{v}, t - \tau \,|\, \mathbf{u}|\mathbf{a}\tau + \mathbf{v}_0|)\, d\tau \tag{4.5}$$

This allows us to relate $P(t; \mathbf{v}_0, n)$ to $f(\mathbf{v}, t \,|\, \mathbf{v}_0)$ by the integral equation

$$f(\mathbf{v}, t \,|\, \mathbf{v}_0) = P(t; \mathbf{v}_0, n)\, \delta(\mathbf{a}t + \mathbf{v}_0 - \mathbf{v}) - \int_0^t d\tau \left[\frac{\partial}{\partial \tau} P(\tau; \mathbf{v}_0, n) \right] \hat{\Pi}_{\mathbf{u}} f(\mathbf{v}, t - \tau \,|\, \mathbf{u}|\mathbf{a}\tau + \mathbf{v}_0|) \tag{4.6}$$

Equation (4.6) will be the basis for the subsequent analysis. Inserting it into Eq. (4.1) and using the relations

$$f(\mathbf{v},0\,|\,\mathbf{u}|\mathbf{a}t+\mathbf{v}_0|) = \delta(\mathbf{v}-\mathbf{u}|\mathbf{a}t+\mathbf{v}_0|) \tag{4.7}$$
$$\hat{\Pi}_{\mathbf{u}}\delta(\mathbf{u}|\mathbf{a}t+\mathbf{v}_0|-\mathbf{v}) = \hat{\Pi}_{\mathbf{v}}\delta(\mathbf{a}t+\mathbf{v}_0-\mathbf{v})$$

we obtain a simple equation satisfied by $P(t;\mathbf{v}_0,n)$

$$\left[\frac{\partial}{\partial t}+\frac{|\mathbf{a}t+\mathbf{v}_0|^n}{2\Lambda}\right]P(t;\mathbf{v}_0,n)=0 \tag{4.8}$$

The physically relevant solution must satisfy the conditions

$$0 \leqslant P(t;\mathbf{v}_0,n) \leqslant 1, \qquad P(0;\mathbf{v}_0,n)=1 \tag{4.9}$$

Note that in one dimension, particles which collide first at time τ are known to have velocity $-(a\tau+v_0)$ after collision, so that Eq. (4.6) should be written in this case as

$$f(v,t\,|\,v_0) = P(t;v_0,n)\,\delta(at+v_0-v) - \int_0^t d\tau\left[\frac{\partial}{\partial\tau}P(\tau;v_0,n)\right]f(v,t-\tau\,|-a\tau-v_0) \tag{4.10}$$

In higher dimensions the distribution of velocity directions is continuous and the unphysical event of a particle having an unchanged direction of motion after collision is included (with weight zero) in Eq. (4.6). Equation (4.8) has, however, the same form in any dimension.

Suppose that the solution of Eq. (4.8), satisfying conditions (4.9) is inserted into Eq. (4.6). An integral equation for $f(\mathbf{v},t\,|\,\mathbf{v}_0)$ is then obtained replacing the initial value problem represented by Eqs. (4.1), (4.2). Both formulations are equivalent.[2]

To begin with, let us suppose that there is no external field ($\mathbf{a}=0$). In this case a simple exponential decay is obtained,

$$P(t;\mathbf{v}_0,n)=\exp(-|\mathbf{v}_0|^n t/2\Lambda) \tag{4.11}$$

with characteristic decay time $\tau_0=2\Lambda|\mathbf{v}_0|^{-n}$. Although for $n<0$, $\tau_0\to\infty$ when $|\mathbf{v}_0|\to\infty$, runaway without collisions is impossible, as for any given n and $\mathbf{v}_0$, $\lim_{t\to\infty}P(t;\mathbf{v}_0,n)=0$.

The situation changes qualitatively when the field is applied to the system. We shall discuss here a number of representative simple cases which suffice to give an overall understanding of the behavior of the gas.

[2] The method of passing to an integral representation of the Boltzmann equation is associated with the name of Chambers[12] in the semiclassical theory of electron motion in metals and semiconductors. We are grateful to Prof. H. L. Frisch for having drawn our attention to this point.

(i) When $n = 0$ we find (in any dimension)

$$P(t; \mathbf{v}_0, n) = \exp(-t/2\Lambda) \tag{4.12}$$

in agreement with the rigorous solution of Section 2.

(ii) Suppose that the initial velocity is directed opposite to the electric field

$$\mathbf{v}_0 = -\mathbf{a}|\mathbf{v}_0|/|\mathbf{a}| \tag{4.13}$$

Equation (4.8) takes the form

$$\left[\frac{\partial}{\partial t} + \frac{1}{2\Lambda}\left||\mathbf{a}|t - |\mathbf{v}_0|\right|^n\right]P(t; \mathbf{v}_0, n) = 0 \tag{4.14}$$

For $n \neq -1$ we find

$$\begin{aligned} P(t; \mathbf{v}_0, n) = &\Big\{\theta(n+1)\theta(|\mathbf{a}|t - |\mathbf{v}_0|)\exp\left[-(|\mathbf{a}|t - |\mathbf{v}_0|)^{n+1}/2|\mathbf{a}|\Lambda(n+1)\right] \\ &+ \theta(|\mathbf{v}_0| - |\mathbf{a}|t)\exp\left[-(|\mathbf{v}_0| - |\mathbf{a}|t)^{n+1}/2|\mathbf{a}|\Lambda(n+1)\right]\Big\} \\ &\times \exp\left[-|\mathbf{v}_0|^{n+1}/2|\mathbf{a}|\Lambda(n+1)\right] \end{aligned} \tag{4.15}$$

whereas when $n = -1$ the solution reads

$$P(t; \mathbf{v}_0, n) = \theta(|\mathbf{v}_0| - |\mathbf{a}|t)\left[1 - \frac{|\mathbf{a}|t}{|\mathbf{v}_0|}\right]^{1/2|\mathbf{a}|\Lambda} \tag{4.16}$$

In all cases $\lim_{t\to\infty} P(t; \mathbf{v}_0, n) = 0$, which means that particles with velocities opposite to the field cannot run away without collisions. The factor $\theta(n+1)$ in the first term on the right-hand side of Eq. (4.15) is to be noted. It implies that when $n < -1$ (the situation where uncollided runaway is expected to occur) the probability of unperturbed motion vanishes after a finite time $|\mathbf{v}_0|/|\mathbf{a}|$. This is so because the probability of suffering a collision increases for slow particles rapidly enough to rule out the possibility of being stopped by the field before a collision takes place. The same holds for $n = -1$ [see Eq. (4.16)].

(iii) Let us now investigate the opposite situation, supposing that particles have an initial velocity oriented along the field

$$\mathbf{v}_0 = \mathbf{a}|\mathbf{v}_0|/|\mathbf{a}| \tag{4.17}$$

Solving Eq. (4.8) yields

$$P(t; \mathbf{v}_0, n) = \begin{cases} \left(1 + \dfrac{|\mathbf{a}|t}{|\mathbf{v}_0|}\right)^{-1/2|\mathbf{a}|\Lambda} & \text{for} \quad n = -1 \\ \exp\left(\dfrac{|\mathbf{v}_0|^{n+1} - (|\mathbf{a}|t + |\mathbf{v}_0|)^{n+1}}{2|\mathbf{a}|\Lambda(n+1)}\right) & \text{for} \quad n \neq -1 \end{cases} \tag{4.18}$$

The fundamental observation about Eq. (4.18) is that whereas for $n \geqslant -1$, $\lim_{t\to\infty} P(t; \mathbf{v}_0, n) = 0$, in the case of $n = -1 - \epsilon$, $\epsilon > 0$, we find

$$\lim_{t\to\infty} P(t; \mathbf{v}_0, -1-\epsilon) = \exp\{-1/2\Lambda|\mathbf{a}|\epsilon|\mathbf{v}_0|^{\epsilon}\} \tag{4.19}$$

Equation (4.19) defines the fraction of particles with initial velocity (4.17) which run away without collisions despite the presence of scattering centers. This fraction vanishes for $|\mathbf{v}_0| \to 0$, and tends to 1 when $|\mathbf{v}_0| \to \infty$. The appearance of uncollided runaway for $n < -1$ is in agreement with a general criterion for this phenomenon discussed in Ref. 5.

(iv) We close the considerations of this section by studying the case of particles moving initially in the direction perpendicular to the field

$$\mathbf{v}_0 \cdot \mathbf{a} = 0 \tag{4.20}$$

Equation (4.20) implies the equality

$$|\mathbf{a}t + \mathbf{v}_0|^n = \left(|\mathbf{a}|^2 t^2 + |\mathbf{v}_0|^2\right)^{n/2} \tag{4.21}$$

so that the solution of Eq. (4.8) takes the form

$$P(t; \mathbf{v}_0, n) = \exp\left\{-\frac{1}{2\Lambda}\int_0^t d\tau \left(|\mathbf{a}|^2\tau^2 + |\mathbf{v}_0|^2\right)^{n/2}\right\} \tag{4.22}$$

The calculation of the long-time limit yields

$$\lim_{t\to\infty} P(t; \mathbf{v}_0, n) = \begin{cases} 0 \quad \text{for} \quad n \geqslant -1 \\ \exp\left\{-\sqrt{\pi}\,\Gamma\left(\frac{\epsilon}{2}\right)/4\Lambda|\mathbf{a}|\,|\mathbf{v}_0|^{\epsilon}\Gamma\left(\frac{1+\epsilon}{2}\right)\right\} \\ \quad \text{for} \quad n = -1-\epsilon, \quad \epsilon > 0 \end{cases} \tag{4.23}$$

We thus find again uncollided runaway for $n < -1$. However, the fraction of particles which escape without collisions is smaller in the case of $\mathbf{v}_0$ perpendicular to $\mathbf{a}$ compared to the situation where $\mathbf{v}_0$ is parallel to $\mathbf{a}$. This immediate consequence of Eqs. (4.19), (4.23) is intuitively clear. Particles moving in the direction of the field have the greatest chance of reaching rapidly the large-velocity region where the collision frequency is negligibly small.

5. DISCUSSION

We considered here charged particles flowing under the action of an electric field through a scattering medium. Particles with velocity $\mathbf{v}$ suffered collisions with frequency proportional to $|\mathbf{v}|^n$. No mechanisms of energy transfer between the gas and scattering centers was present. Our results combined with those obtained in Ref. 9 yield a rather complete picture of qualitative changes in the behavior of the system when the exponent n is varied.

For $n > 0$ the mean velocity of the gas vanishes in the long-time limit: the runaway phenomenon does not occur. The effect of the field shows up, however, in the unlimited growth of the kinetic energy $\sim t^{2/(n+2)}$. This long-time behavior, established for $n > -1$ in Ref. 9, reflects the most unsatisfactory feature of the Lorentz model studied here. The gas is heated indefinitely by the field because of the lack of energy losses at encounters with immobile scatterers. For no value of n does there exist a stationary regime with finite thermal energy density.

When $n = 0$ the gas starts moving as a whole: its mean velocity attains exponentially a constant nonzero value. For collision frequencies tending to zero in the large-velocity limit ($n < 0$) a new phenomenon, called runaway, arises: the applied field increases the mean velocity of flowing particles without bounds. For n not too small ($-1 < n < 0$) this growth of velocity is governed by the power law $t^{-n/(n+2)}$. Collisions remain important in the sense that the dominant term in the long-time expansion of the velocity distribution shows spherical symmetry.

For $n = -1$ linear growth of the mean gas velocity is attained and the field forces an anisotropy in the velocity distribution that persists for arbitrary times.

Finally, a spectacular effect appears when the collision frequency decreases faster than $|\mathbf{v}|^{-1}$ for $|\mathbf{v}| \to \infty$. A finite fraction of particles then moves without collisions, freely accelerated by the field. The occurrence of such uncollided runaway for $n < -1$ was predicted in Ref. 5.

The long-time expansion of Ref. 9 [see Eq. (59)] seems also to assign a special character to the case $n = -2$. We found that in one dimension the singularity $|v|^{-2}$ of the collision frequency at $v = 0$ excludes the existence of a solution to the initial value problem $f(v, t = 0) = \delta(v)$. In order to see this, note that the homogeneous equation associated with Eq. (2.8) takes the form of Whittaker equation (3.3) with $\lambda = 0$, $\mu = (1 + 4z/\Lambda a^2)^{1/2}/2$ when $n = -2$, $v > 0$. It is then readily checked that the behavior

$$W_{0,\mu}(2vz/a) \underset{v \to 0}{\sim} v^{(1/2)-\mu} \tag{5.1}$$

is incompatible with condition (3.5). In this sense one could say that for $n = -2$ (and probably for $n < -2$) collisions exclude the possibility of zero velocity.

Before closing, let us consider the possible applicability of conclusions derived for the Lorentz gas to more realistic systems. In Section 1 we indicated that the collision frequency $\sim |\mathbf{v}|^n$ corresponds to scattering in the potential field $\sim |\mathbf{r}|^{-4/(1-n)}$: Hence runaway should occur for potentials decreasing more slowly than $|\mathbf{r}|^{-4}$ at large distances ($n < 0$). This includes of course the Coulomb case. We think that this conclusion might be valid for more realistic models. The fact that we used the hard-sphere angle dependence of the cross section does not seem to be very important from this point of view. For instance, it is clear that the argument presented in Section 4 to calculate the probability of collisionless motion does not depend on the fact that the projector $\hat{\Pi}_v$ assigns equal weight to all directions. The unphysical character of the Lorentz model corresponds rather to the lack of energy transfer between flowing particles and the scattering medium. No thermalization of the gas is possible in these conditions. This makes the situation quite different from that considered in plasma physics, where a critical field is required to accelerate effectively the electrons despite their losses of energy at encounters with the ions (see, e.g., Ref. 3). Let us also remark that the notion of uncollided runaway is an idealization of what could happen in a realistic system with $n < -1$. Indeed, particles moving through a medium exerting long-range forces on them are permanently influenced by scatterers. So the probability of collisionless motion is strictly speaking zero at any time.

We stressed in our analysis that the growing influence of the field for decreasing n shows up for $n = -1$ in the breaking of the asymptotic spherical symmetry of the velocity distribution. Of course, in a realistic description of scattering the collision-dominated shape of the velocity distribution should correspond to the actual angular dependence of the cross section. The spherical symmetry obtained in our considerations for $n > -1$ was due to the assumed hard-sphere isotropic scattering. However, the fact that at $n = -1$ the field introduces its own anisotropy into the velocity distribution may remain true in a more realistic description of the flow of the gas.

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